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The Cosserat spectrum for cylindrical geometries (Part 1: discrete subspace)

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Abstract

By directly solving the Navier equations of elasticity, we obtain the discrete Cosserat eigenvalues and eigenvectors for the first boundary value problem of a cylindrical shell. The discrete Cosserat spectrum approaches $\tilde{\omega}_n = -2$ from both $\tilde{\omega}_n < -2$ and $\tilde{\omega}_n > -2$ sides. It also reduces to a condensation point $\tilde{\omega}_n = -2$ with infinite multiplicity for a cylinder or a cylindrical rigid inclusion in an infinite space.

1. Introduction

Cosserat and Cosserat (1898) showed that in a domain V the homogeneous Navier equations under homogeneous boundary displacement

$$\Delta \boldsymbol{u} + \omega \nabla\nabla \cdot \boldsymbol{u} = \boldsymbol{0} \quad \text{in } V \tag{1a}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial V \tag{1b}$$

where $\omega = (\lambda + \mu)/\mu = 1/(1 - 2v)$, λ and μ are the Lamé constants, v the Poisson's ratio and ∂V the boundary of V, admit non-trivial solution when ω takes some special values $\tilde{\omega}$. Obviously the Cosserat spectrum $\tilde{\omega}$ lies outside the range of the uniqueness theory in elasticity (Knops and Payne, 1971). The Cosserat spectrum theory was fully developed by Mikhlin (1973) who proved the completeness and orthogonality of the Cosserat eigenfunctions and represented the displacement field **u** for an inhomogeneous problem as a summation of the Cosserat eigenfunctions.

For the first boundary value problem (boundary value problem of displacement) in 3-D, the

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eigenvectors are complete in the Hilbert space H^1 and form three orthogonal subspaces, namely, the discrete eigenvectors $\tilde{\boldsymbol{u}}_n$, the eigenvectors $\tilde{\boldsymbol{u}}_n^{(-1)}$ corresponding to the eigenvalue of infinite multiplicity $\tilde{\omega} = -1$ and the eigenvectors $\tilde{\boldsymbol{u}}_n^{(\infty)}$ corresponding to the eigenvalue of infinite multiplicity $\tilde{\omega} = -\infty$. The solution of the inhomogeneous problem

$$\Delta \boldsymbol{u} + \omega \nabla \nabla \cdot \boldsymbol{u} = -\frac{\boldsymbol{F}}{\mu} \quad \text{in } V \tag{2a}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial V \tag{2b}$$

admits the representation (Mikhlin, 1973)

$$\boldsymbol{u} = \sum_n \left\{ \frac{\left(\boldsymbol{f}, \tilde{\boldsymbol{u}}_n^{(-1)}\right)}{1+\omega} \tilde{\boldsymbol{u}}_n^{(-1)} + \left(\boldsymbol{f}, \tilde{\boldsymbol{u}}_n^{(\infty)}\right) \tilde{\boldsymbol{u}}_n^{(\infty)} + \frac{\tilde{\omega}_n}{\tilde{\omega}_n - \omega} (\boldsymbol{f}, \tilde{\boldsymbol{u}}_n) \tilde{\boldsymbol{u}}_n \right\} \tag{3a}$$

where

$$(\boldsymbol{f}, \tilde{\boldsymbol{u}}) \equiv \frac{1}{\mu} \int_V \boldsymbol{F} \cdot \tilde{\boldsymbol{u}} \, \mathrm{d}V \tag{3b}$$

and $\mathrm{d}V$ is the volume element in 3-D.

The solution of the problem with inhomogeneous boundary displacement

$$\Delta \boldsymbol{u} + \omega \nabla \nabla \cdot \boldsymbol{u} = \boldsymbol{0} \quad \text{in } V \tag{4a}$$

$$\boldsymbol{u} = \boldsymbol{u}_b \quad \text{on } \partial V \tag{4b}$$

admits the representation (Mikhlin, 1973)

$$\boldsymbol{u} = \boldsymbol{u}_0 + \sum_n \frac{\omega \tilde{\omega}_n}{\omega - \tilde{\omega}_n} (\operatorname{div} \boldsymbol{u}_0, \operatorname{div} \tilde{\boldsymbol{u}}_n) \tilde{\boldsymbol{u}}_n \tag{5}$$

where $\boldsymbol{u}_0$ is a vector harmonic function and satisfies the boundary condition eqn (4b).

For the secondary boundary value problem (boundary value problem of traction) in 3-D, the eigenvectors are complete in the Hilbert space H^1 and form three orthogonal subspaces, namely $\tilde{\boldsymbol{u}}_n, \tilde{\boldsymbol{u}}_n^{(-1)}$ and $\tilde{\boldsymbol{u}}_n^{(\infty)}$. The solution of the inhomogeneous problem

$$\Delta \boldsymbol{u} + \omega \nabla \nabla \cdot \boldsymbol{u} = -\frac{\boldsymbol{F}}{\mu} \quad \text{in } V \tag{6a}$$

$$\boldsymbol{t} = \boldsymbol{t}_b \quad \text{on } \partial V \tag{6b}$$

admits the representation (Mikhlin, 1973)

$$\boldsymbol{u} = \sum_n \left\{ \frac{2\left(\boldsymbol{f}, \tilde{\boldsymbol{u}}_n^{(-1)}\right)}{1+\omega} \tilde{\boldsymbol{u}}_n^{(-1)} + \left(\boldsymbol{f}, \tilde{\boldsymbol{u}}_n^{(\infty)}\right) \tilde{\boldsymbol{u}}_n^{(\infty)} + \frac{1-\tilde{\omega}_n}{\omega - \tilde{\omega}_n} (\boldsymbol{f}, \tilde{\boldsymbol{u}}_n) \tilde{\boldsymbol{u}}_n \right\} \tag{7a}$$

where

$$\left(\boldsymbol{f},\tilde{\boldsymbol{u}}\right)\equiv\frac{1}{\mu}\left[\int_V \boldsymbol{F}\cdot\tilde{\boldsymbol{u}}\,\mathrm{d}V+\int_{\partial V}\boldsymbol{t}_b\cdot\tilde{\boldsymbol{u}}\,\mathrm{d}A\right] \tag{7b}$$

and $\mathrm{d}V$ and $\mathrm{d}A$ are the volume and surface elements in 3-D, respectively.

The representation theorems for 2-D elastic problems take simpler forms. For the first boundary value problem in a 2-D domain A, the Cosserat spectrum consists of two isolated points, $\tilde{\omega}=-\infty,\,-1$, and a discrete spectrum $\tilde{\omega}_n$. In some cases the discrete spectrum may reduce to a condensation point $\tilde{\omega}_n=-2$ with infinite multiplicity. The system of all the Cosserat eigenvectors is orthogonal and complete in the Hilbert space $H^{1,\varepsilon}$ in which

$$\int_A \frac{(\nabla\cdot\tilde{\boldsymbol{u}})^2}{r^\varepsilon}\,\mathrm{d}A<\infty \quad \text{for all } \varepsilon>0 \tag{8}$$

where $\mathrm{d}A$ is the area element in 2-D. The boundary value problem of displacement

$$\Delta\boldsymbol{u}+\omega\nabla\nabla\cdot\boldsymbol{u}=-\frac{\boldsymbol{F}}{\mu} \quad \text{in } A \tag{9a}$$

$$\boldsymbol{u}=\boldsymbol{0} \quad \text{on } \partial A \tag{9b}$$

admits the representation eqn (3a) while the inner product is defined by

$$\left(\boldsymbol{f},\tilde{\boldsymbol{u}}\right)\equiv\frac{1}{\mu}\int_A \boldsymbol{F}\cdot\tilde{\boldsymbol{u}}\,\mathrm{d}A \tag{10}$$

For the second boundary value problem in 2-D, the Cosserat spectrum consists of only three points, $\tilde{\omega}=-\infty,\,-1,\,0$. The system of the Cosserat eigenvectors is orthogonal and complete in the Hilbert space $H^{1,\varepsilon}$. It follows that the solution of the second boundary value problem in 2-D

$$\Delta\boldsymbol{u}+\omega\nabla\nabla\cdot\boldsymbol{u}=-\frac{\boldsymbol{F}}{\mu} \quad \text{in } A \tag{11a}$$

$$\boldsymbol{t}=\boldsymbol{t}_b \quad \text{on } \partial A \tag{11b}$$

admits the representation

$$\boldsymbol{u}=\sum_n\left\{\frac{2\left(\boldsymbol{f},\tilde{\boldsymbol{u}}_n^{(-1)}\right)}{1+\omega}\tilde{\boldsymbol{u}}_n^{(-1)}+\left(\boldsymbol{f},\tilde{\boldsymbol{u}}_n^{(\infty)}\right)\tilde{\boldsymbol{u}}_n^{(\infty)}+\frac{1}{\omega}\left(\boldsymbol{f},\tilde{\boldsymbol{u}}_n^{(0)}\right)\tilde{\boldsymbol{u}}_n^{(0)}\right\} \tag{12a}$$

where

$$\left(\boldsymbol{f},\tilde{\boldsymbol{u}}\right)\equiv\frac{1}{\mu}\left[\int_A \boldsymbol{F}\cdot\tilde{\boldsymbol{u}}\,\mathrm{d}A+\int_{\partial A}\boldsymbol{t}_b\cdot\tilde{\boldsymbol{u}}\,\mathrm{d}S\right] \tag{12b}$$

and $\mathrm{d}A$ and $\mathrm{d}S$ are the elements of A and ∂A, respectively.

Thus, knowing the Cosserat eigenvectors for a given geometry allows us to solve elastic problems for any body force and boundary loading. Markenscoff and Paukshto (1998), Liu et al. (1998), Markenscoff et al. (1998) have applied the Cosserat spectrum theory to elasticity, thermoelasticity and viscoelasticity problems.

The Cosserat eigenvectors may not all appear in a representation for a specific loading. For example, we only need to know the discrete Cosserat eigenvectors $\tilde{\boldsymbol{u}}_n$ for a harmonic temperature loading in thermoelasticity, while we need both $\tilde{\boldsymbol{u}}_n$ and $\tilde{\boldsymbol{u}}^{(-1)}$ for a non-harmonic temperature loading (Liu et al., 1998). In order to solve 2-D problems with cylindrical body shapes, we need to find the pertinent Cosserat eigenvalues and eigenvectors. For this purpose, in the first part of this paper, we solve the 2-D Navier equations and obtain the discrete Cosserat eigenvalues and eigenvectors for the first boundary value problem of a cylindrical shell. These eigenvalues and eigenvectors reduce to those for a solid cylinder (inner problem) and a cylindrical rigid inclusion (outer problem) in an infinite space. In the second part, we will present the Cosserat subspace $\tilde{\boldsymbol{u}}^{(-1)}$ for a solid cylinder and a cylindrical rigid inclusion. We also present an example of a non-harmonic heat source in the presence of a cylindrical rigid inclusion. The results show that the sequence of the Cosserat eigenvectors $\tilde{\boldsymbol{u}}^{(-1)}$ converges fast, thus providing a practical way to solve problems for general body force and boundary loading.

2. A cylindrical shell

The discrete Cosserat eigenvalue $\tilde{\omega}_n$ and eigenvector $\tilde{\boldsymbol{u}}_n$ for the first boundary value problem satisfy the homogeneous Navier equations with homogeneous boundary conditions

$$\Delta\tilde{\boldsymbol{u}}_n + \tilde{\omega}_n \nabla\nabla\cdot\tilde{\boldsymbol{u}}_n = \mathbf{0} \quad \text{in } A \tag{13a}$$

$$\tilde{\boldsymbol{u}}_n = \mathbf{0} \quad \text{on } \partial A \tag{13b}$$

Taking the divergence on eqn (13a) yields

$$(1+\tilde{\omega}_n)\Delta(\nabla\cdot\tilde{\boldsymbol{u}}_n) = 0 \tag{14}$$

Equation (14) means that, since the discrete Cosserat eigenvalue $\tilde{\omega}_n \neq -1$, $\nabla\cdot\tilde{\boldsymbol{u}}_n$ is any harmonic function. For 2-D cylindrical bodies, the complete collection of harmonic functions in a polar coordinate system $(r,\ \theta)$ is (Zachmanoglou and Thoe, 1976)

$$\nabla\cdot\tilde{\boldsymbol{u}}_n = \begin{cases} 1,\ r^n\cos n\theta, \quad r^n \sin n\theta \\ \\ \log r,\ r^{-n}\cos n\theta, \quad r^{-n}\sin n\theta \end{cases} \qquad n = 1,\ 2,\ldots \tag{15}$$

For a specific geometry, one needs to choose appropriate harmonic functions from eqn (15) which satisfy the requirement of the Hilbert space $H^{1,\varepsilon}$. The appropriate $\tilde{\boldsymbol{u}}_n$ is then substituted back into eqn (13) to solve for $\tilde{\omega}_n$ and $\tilde{\boldsymbol{u}}_n$. The Cosserat eigenvector $\tilde{\boldsymbol{u}}_n$ of the first boundary value problem in 2-D should also be normalized according to (Mikhlin, 1973)

$$\int_A (\nabla\cdot\tilde{\boldsymbol{u}}_n)^2\,\mathrm{d}A = -\frac{1}{\tilde{\omega}_n} \tag{16}$$

We consider a cylindrical shell $r_1 \leqslant r \leqslant r_2$, where r_1 and r_2 are the inner and outer radius of the cylindrical shell, respectively. All the harmonic functions represented by eqn (15) satisfy the requirement in the Hilbert space $H^{1,\varepsilon}$.

Write the Cosserat eigenvector $\tilde{\boldsymbol{u}}_n$ in the form of separation of variables

$$\tilde{\boldsymbol{u}}_n = u_{nr}e_r + u_{n\theta}e_\theta = R_{1n}(r)Q_{1n}(\theta)e_r + R_{2n}(r)Q_{2n}(\theta)e_\theta \tag{17}$$

The boundary conditions eqn (13b) now take the form

$$R_{in}(r_1) = R_{in}(r_2) = 0 \quad i = 1, 2 \tag{18}$$

According to eqn (14), $\nabla \cdot \tilde{\boldsymbol{u}}_n$ must be a general harmonic function, thus, it must be of the form

$$\nabla \cdot \tilde{\boldsymbol{u}}_n = \left[A_n \left(\frac{r}{r_2} \right)^n + B_n \left(\frac{r_1}{r} \right)^n \right] \cos n\theta \tag{19}$$

Substituting eqn (17) into eqn (19) yields

$$\nabla \cdot \tilde{\boldsymbol{u}}_n = \frac{\mathrm{d}R_{1n}}{\mathrm{d}r} Q_{1n} + \frac{R_{1n}}{r} Q_{1n} + \frac{R_{2n}}{r} \frac{\mathrm{d}Q_{2n}}{\mathrm{d}\theta} = \left[A_n \left(\frac{r}{r_2} \right)^n + B_n \left(\frac{r_1}{r} \right)^n \right] \cos n\theta \tag{20}$$

To make the variables separable, we choose $Q_{1n} = \cos n\theta$ and $Q_{2n} = \sin n\theta$. Consequently, eqn (20) becomes

$$\frac{\mathrm{d}R_{1n}}{\mathrm{d}r} + \frac{R_{1n}}{r} + \frac{nR_{2n}}{r} = A_n \left(\frac{r}{r_2} \right)^n + B_n \left(\frac{r_1}{r} \right)^n \tag{21}$$

Equation (17) should also satisfy the Navier equations, eqn (13a), which now take the following forms

$$\frac{\mathrm{d}^2 R_{1n}}{\mathrm{d}r^2} + \frac{1}{r} \frac{\mathrm{d}R_{1n}}{\mathrm{d}r} - \frac{(n^2+1) R_{1n}}{r^2} - \frac{2nR_{2n}}{r^2} + n\tilde{\omega}_n \left(A_n \frac{r^{n-1}}{r_2^n} - B_n \frac{r_1^n}{r^{n+1}} \right) = 0 \tag{22a}$$

$$\frac{\mathrm{d}^2 R_{2n}}{\mathrm{d}r^2} + \frac{1}{r} \frac{\mathrm{d}R_{2n}}{\mathrm{d}r} - \frac{(n^2+1) R_{2n}}{r^2} - \frac{2nR_{1n}}{r^2} - n\tilde{\omega}_n \left(A_n \frac{r^{n-1}}{r_2^n} + B_n \frac{r_1^n}{r^{n+1}} \right) = 0 \tag{22b}$$

The sum and difference of eqns (22a) and (22b) give

$$\frac{\mathrm{d}^2(R_{1n}+R_{2n})}{\mathrm{d}r^2} + \frac{1}{r} \frac{\mathrm{d}(R_{1n}+R_{2n})}{\mathrm{d}r} - \frac{(n+1)^2(R_{1n}+R_{2n})}{r^2} - \frac{2n\tilde{\omega}_n B_n r_1^n}{r^{n+1}} = 0 \tag{23}$$

$$\frac{\mathrm{d}^2(R_{1n}-R_{2n})}{\mathrm{d}r^2} + \frac{1}{r} \frac{\mathrm{d}(R_{1n}-R_{2n})}{\mathrm{d}r} - \frac{(n-1)^2(R_{1n}-R_{2n})}{r^2} + \frac{2n\tilde{\omega}_n A_n r^{n-1}}{r_2^n} = 0 \tag{24}$$

The solution of eqn (23) under boundary condition eqn (18) is given by

$$R_{1n} + R_{2n} = \frac{1}{2} \tilde{\omega}_n B_n \left(C_{1n} \frac{r^{n+1}}{r_2^n} + C_{2n} \frac{r_1^{n+2}}{r^{n+1}} - \frac{r_1^n}{r^{n-1}} \right) \tag{25}$$

where

$$C_{1n} = (r_1/r_2)^n \left[1 - (r_1/r_2)^2 \right] / \left[1 - (r_1/r_2)^{2(n+1)} \right] \tag{26a}$$

$$C_{2n} = \left[1 - (r_1/r_2)^{2n} \right] / \left[1 - (r_1/r_2)^{2(n+1)} \right] \tag{26b}$$

The solution of eqn (24) under boundary condition eqn (18) is given by

$$R_{11} - R_{21} = \frac{1}{2}\tilde{\omega}_1 A_1\left(C_{31}r_2 - C_{41}r_2\log\left(\frac{r}{r_2}\right) - \frac{r^2}{r_2}\right) \tag{27a}$$

$$R_{1n} - R_{2n} = \frac{1}{2}\tilde{\omega}_n A_n\left(C_{3n}\frac{r^{n-1}}{r_2^{n-2}} + C_{4n}\frac{r_1^n}{r^{n-1}} - \frac{r^{n+1}}{r_2^n}\right) \quad n > 1 \tag{27b}$$

where

$$C_{31} = 1 \tag{28a}$$

$$C_{41} = \left[1 - (r_1/r_2)^2\right]/\log(r_1/r_2) \tag{28b}$$

$$C_{3n} = \left[1 - (r_1/r_2)^{2n}\right]/\left[1 - (r_1/r_2)^{2(n-1)}\right] \quad n > 1 \tag{28c}$$

$$C_{4n} = (r_1/r_2)^n\left[1 - (r_1/r_2)^{-2}\right]/\left[1 - (r_1/r_2)^{2(n-1)}\right] \quad n > 1 \tag{28d}$$

Simple algebraic operations on eqns (25) and (27) give

$$R_{i1} = \frac{\tilde{\omega}_1 B_1}{4}\left(C_{11}\frac{r^2}{r_2} + C_{21}\frac{r_1^3}{r^2} - r_1\right) \pm \frac{\tilde{\omega}_1 A_1}{4}\left(C_{31}r_2 - C_{41}r_2\log\frac{r}{r_2} - \frac{r^2}{r_2}\right) \quad i = 1, 2 \tag{29a}$$

$$\begin{aligned} R_{in} = &\frac{\tilde{\omega}_n B_n}{4}\left(C_{1n}\frac{r^{n+1}}{r_2^n} + C_{2n}\frac{r_1^{n+2}}{r^{n+1}} - \frac{r_1^n}{r^{n-1}}\right) \\ &\pm \frac{\tilde{\omega}_n A_n}{4}\left(C_{3n}\frac{r^{n-1}}{r_2^{n-2}} + C_{4n}\frac{r_1^n}{r^{n-1}} - \frac{r^{n+1}}{r_2^n}\right) \quad n > 1, \;\; i = 1, 2 \end{aligned} \tag{29b}$$

where the positive and negative sign apply when $i = 1$ and $i = 2$, respectively. We now obtain the Cosserat eigenvector $\tilde{\boldsymbol{u}}_n = u_{nr}e_r + u_{n\theta}e_\theta$ as follows

$$\begin{Bmatrix} u_{1r} \\ u_{1\theta} \end{Bmatrix} = \frac{\tilde{\omega}_1}{4}\left[B_1\left(C_{11}\frac{r^2}{r_2} + C_{21}\frac{r_1^3}{r^2} - r_1\right) \pm A_1\left(C_{31}r_2 - C_{41}r_2\log\frac{r}{r_2} - \frac{r^2}{r_2}\right)\right]\begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix} \tag{30a}$$

$$\begin{aligned} \begin{Bmatrix} u_{nr} \\ u_{n\theta} \end{Bmatrix} = &\frac{\tilde{\omega}_n}{4}\left[B_n\left(C_{1n}\frac{r^{n+1}}{r_2^n} + C_{2n}\frac{r_1^{n+2}}{r^{n+1}} - \frac{r_1^n}{r^{n-1}}\right)\right. \\ &\left.\pm A_n\left(C_{3n}\frac{r^{n-1}}{r_2^{n-2}} + C_{4n}\frac{r_1^n}{r^{n-1}} - \frac{r^{n+1}}{r_2^n}\right)\right]\begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} \quad n > 1 \end{aligned} \tag{30b}$$

where the positive and negative sign apply to u_{nr} and $u_{n\theta}$, respectively.

To find the Cosserat eigenvalue, we substitute eqn (29) into eqn (21)

$$\frac{\tilde{\omega}_1 B_1}{2}\left(2C_{11}\frac{r}{r_2}-\frac{r_1}{r}\right)-\frac{\tilde{\omega}_1 A_1}{4}\left(C_{41}\frac{r_2}{r}+\frac{2r}{r_2}\right)=A_1\frac{r}{r_2}+B_1\frac{r_1}{r} \tag{31a}$$

$$\frac{\tilde{\omega}_n B_n}{2}\left[(n+1)C_{1n}\frac{r^n}{r_2^n}-\frac{r_1^n}{r^n}\right]-\frac{\tilde{\omega}_n A_n}{2}\left[(n-1)C_{4n}\frac{r_1^n}{r^n}+\frac{r^n}{r_2^n}\right]=A_n\frac{r^n}{r_2^n}+B_n\frac{r_1^n}{r^n}\quad n>1 \tag{31b}$$

Equating the coefficients of the same power of r in eqn (31), we have

$$-\frac{r_2}{2r_1}\tilde{\omega}_1 C_{41}A_1=(\tilde{\omega}_1+2)B_1 \tag{32a}$$

$$(\tilde{\omega}_1+2)A_1=2\tilde{\omega}_1 C_{11}B_1 \tag{32b}$$

$$-(n-1)\tilde{\omega}_n C_{4n}A_n=(\tilde{\omega}_n+2)B_n\quad n>1 \tag{32c}$$

$$(\tilde{\omega}_n+2)A_n=(n+1)\tilde{\omega}_n C_{1n}B_n\quad n>1 \tag{32d}$$

Equation (32) gives the Cosserat eigenvalue $\tilde{\omega}_n$ and A_n/B_n as follows

$$(\tilde{\omega}_1+2)^2=-\tilde{\omega}_1^2\frac{1-(r_1/r_2)^2}{\left[1+(r_1/r_2)^2\right]\log(r_1/r_2)} \tag{33a}$$

$$(\tilde{\omega}_n+2)^2=(n^2-1)\tilde{\omega}_n^2\frac{(r_1/r_2)^{2n}\left[(r_1/r_2)^2+(r_1/r_2)^{-2}-2\right]}{\left[1-(r_1/r_2)^{2(n+1)}\right]\left[1-(r_1/r_2)^{2(n-1)}\right]}\quad n>1 \tag{33b}$$

$$\left(\frac{A_1}{B_1}\right)^2=-\frac{4(r_1/r_2)^2\log(r_1/r_2)}{1-(r_1/r_2)^4} \tag{34a}$$

$$\left(\frac{A_n}{B_n}\right)^2=\frac{n+1}{n-1}\frac{(r_1/r_2)^2\left[1-(r_1/r_2)^{2(n-1)}\right]}{1-(r_1/r_2)^{2(n+1)}}\quad n>1 \tag{34b}$$

To determine the coefficients A_n and B_n, we also need to use the normalization condition eqn (16). By substituting eqn (19) into eqn (16), we have

$$\frac{A_1^2r_2^2}{4}\left[1-\left(\frac{r_1}{r_2}\right)^4\right]+A_1B_1\frac{r_1}{r_2}\left(r_2^2-r_1^2\right)-B_1^2r_1^2\log\left(\frac{r_1}{r_2}\right)=-\frac{1}{\pi\tilde{\omega}_1} \tag{35a}$$

$$\frac{A_n^2r_2^2}{2(n+1)}\left[1-(r_1/r_2)^{2(n+1)}\right]+A_nB_n(r_1/r_2)^n\left(r_2^2-r_1^2\right)+\frac{B_n^2r_1^2}{2(n-1)}\left[1-(r_1/r_2)^{2(n-1)}\right]=-\frac{1}{\pi\tilde{\omega}_n}\quad n>1 \tag{35b}$$

To derive the second series of the Cosserat eigenvalues and eigenvectors, we choose from eqn (15) the harmonic function

$$\nabla \cdot \tilde{\boldsymbol{u}}_n = \left[A_n \left(\frac{r}{r_2} \right)^n + B_n \left(\frac{r_1}{r} \right)^n \right] \sin n\theta \tag{36}$$

where $n = 1, 2, 3, \ldots$. Proceeding in a similar manner, we obtain the discrete Cosserat eigenvalues $\tilde{\omega}_n$ and eigenvectors $\tilde{\boldsymbol{u}}_n = u_{nr} e_r + u_{n\theta} e_\theta$ as follows

$$(\tilde{\omega}_1 + 2)^2 = -\tilde{\omega}_1^2 \frac{1 - (r_1/r_2)^2}{\left[1 + (r_1/r_2)^2\right] \log(r_1/r_2)} \tag{37a}$$

$$(\tilde{\omega}_n + 2)^2 = (n^2 - 1)\tilde{\omega}_n^2 \frac{(r_1/r_2)^{2n}\left[(r_1/r_2)^2 + (r_1/r_2)^{-2} - 2\right]}{\left[1 - (r_1/r_2)^{2(n+1)}\right]\left[1 - (r_1/r_2)2^{(n-1)}\right]} \quad n > 1 \tag{37b}$$

$$\begin{Bmatrix} u_{1r} \\ u_{1\theta} \end{Bmatrix} = \frac{\tilde{\omega}_1}{4} \left[A_1 \left(C_{31} r_2 - C_{41} r_2 \log \frac{r}{r_2} - \frac{r^2}{r_2} \right) \pm B_1 \left(C_{11} \frac{r^2}{r_2} + C_{21} \frac{r_1^3}{r^2} - r_1 \right) \right] \begin{Bmatrix} \sin\theta \\ \cos\theta \end{Bmatrix} \tag{38a}$$

$$\begin{Bmatrix} u_{nr} \\ u_{n\theta} \end{Bmatrix} = \frac{\tilde{\omega}_n}{4} \left[A_n \left(C_{3n} \frac{r^{n-1}}{r_2^{n-2}} + C_{4n} \frac{r_1^n}{r^{n-1}} - \frac{r^{n+1}}{r_2^n} \right) \right. \\ \left. \pm B_n \left(C_{1n} \frac{r^{n+1}}{r_2^n} + C_{2n} \frac{r_1^{n+2}}{r^{n+1}} - \frac{r_1^n}{r^{n-1}} \right) \right] \begin{Bmatrix} \sin n\theta \\ \cos n\theta \end{Bmatrix} \quad n > 1 \tag{38b}$$

where the coefficients C_{in}, $i = 1, \ldots, 4$, are still given by eqns (26) and (28), A_n and B_n are still given by eqns (34) and (35).

To derive the third Cosserat eigenvalue and eigenvector, we repeat the above procedures with

$$\nabla \cdot \tilde{\boldsymbol{u}} = A + B \log \frac{r}{r_2} \tag{39}$$

We write $\tilde{\boldsymbol{u}}$ in the form of separated variables

$$\tilde{\boldsymbol{u}} = u_r e_r + u_\theta e_\theta = R_1(r) Q_1(\theta) e_r + R_2(r) Q_2(\theta) e_\theta \tag{40}$$

The boundary conditions are now expressed by

$$R_i(r_1) = R_i(r_2) = 0 \quad i = 1, 2 \tag{41}$$

Substituting eqn (40) into eqn (39), we have

$$\left(\frac{\mathrm{d}R_1}{\mathrm{d}r} + \frac{R_1}{r} \right) Q_1 + \frac{R_2}{r} \frac{\mathrm{d}Q_2}{\mathrm{d}\theta} = A + B \log \frac{r}{r_2} \tag{42}$$

Eqn (42) shows that, in order to make the variables separable, we need to choose $Q_1 = 1$ and $(\mathrm{d}Q_2/\mathrm{d}\theta) = 1$ or $Q_2 = \theta + c$, where c is an arbitrary constant. Consequently, eqn (42) becomes

$$\frac{\mathrm{d}R_1}{\mathrm{d}r} + \frac{R_1}{r} + \frac{R_2}{r} = A + B \log \frac{r}{r_2} \tag{43}$$

Eqn (40) should also satisfy the Navier equations, eqn (13a), which now take the form

$$\frac{d^2 R_1}{dr^2} + \frac{1}{r}\frac{dR_1}{dr} - \frac{R_1}{r^2} - \frac{2R_2}{r^2} + \frac{\tilde{\omega} B}{r} = 0 \tag{44a}$$

$$\frac{d^2 R_2}{dr^2} + \frac{1}{r}\frac{dR_2}{dr} - \frac{R_2}{r^2} = 0 \tag{44b}$$

The solution of eqn (44) subjected to boundary condition eqn (41) is

$$R_1 = C_1 r + C_2 \frac{r_1^2}{r} + \frac{\tilde{\omega} B}{4} r - \frac{\tilde{\omega} B}{2} r \log r \tag{45a}$$

$$R_2 = 0 \tag{45b}$$

where

$$C_1 = \frac{\tilde{\omega} B}{4}\left[2 \log r_1 - \frac{2 \log(r_1/r_2)}{1 - (r_1/r_2)^2} - 1\right] \tag{46a}$$

$$C_2 = \frac{\tilde{\omega} B \log(r_1/r_2)}{2\left[1 - (r_1/r_2)^2\right]} \tag{46b}$$

Substituting eqn (45) into eqn (40), we obtain the Cosserat eigenvector

$$\tilde{\boldsymbol{u}} = \left[C_1 r + C_2 \frac{r_1^2}{r} + \frac{\tilde{\omega} B}{4} r - \frac{\tilde{\omega} B}{2} r \log r\right] e_r \tag{47}$$

To find the Cosserat eigenvalue, substituting eqn (45) into eqn (43), we have

$$(A - B \log r_2) + (\tilde{\omega} + 1) B \log r = 0 \tag{48}$$

Equating the coefficients of eqn (48), we obtain the Cosserat eigenvalue $\tilde{\omega}$ and the ratio A/B

$$\tilde{\omega} = -1 \tag{49}$$

$$A/B = \log r_2 \tag{50}$$

The coefficients A and B are also subjected to the normalization condition eqn (16). Substituting eqn (50) into eqn (39) yields $\nabla \cdot \tilde{\boldsymbol{u}} = B \log r$, which is substituted into eqn (16), we have

$$r_2^2\left[\log^2 r_2 - \log r_2 + 1/2\right] - r_1^2\left[\log^2 r_1 - \log r_1 + 1/2\right] = 1/(\pi B) \tag{51}$$

In two limiting cases, the discrete Cosserat eigenvalues $\tilde{\omega}_n$ and eigenvectors $\tilde{\boldsymbol{u}}_n$ for the first boundary value problem of a cylindrical shell reduce to those for a solid cylinder and a cylindrical rigid inclusion in an infinite space, which are presented in the next two sections.

3. A solid cylinder

If $r_1 = 0$ and $r_2 = r_0$, the cylindrical shell reduces to a solid cylinder with radius r_0. Since $\tilde{\boldsymbol{u}}_n$ has to satisfy the requirement in the Hilbert space $H^{1,\varepsilon}$, only the functions 1, $r^n \cos n\theta, r^n \sin n\theta$ need to be considered. The sets of the discrete Cosserat eigenvalues $\tilde{\omega}_n$, eigenvectors $\tilde{\boldsymbol{u}}_n$ and divergences div $\tilde{\boldsymbol{u}}_n$ are

$$\tilde{\omega}_n = -2 \tag{52}$$

$$\tilde{\boldsymbol{u}}_n = \frac{A_n}{2}\frac{r^{n+1}}{r_0^n}\left(1 - \frac{r_0^2}{r^2}\right)(\cos n\theta e_r - \sin n\theta e_\theta) \tag{53}$$

$$\nabla \cdot \tilde{\boldsymbol{u}}_n = A_n\left(\frac{r}{r_0}\right)^n \cos n\theta \tag{54}$$

and

$$\tilde{\omega}_n = -2 \tag{55}$$

$$\tilde{\boldsymbol{u}}_n = \frac{A_n}{2}\frac{r^{n+1}}{r_0^n}\left(1 - \frac{r_0^2}{r^2}\right)(\sin n\theta e_r + \cos n\theta e_\theta) \tag{56}$$

$$\nabla \cdot \tilde{\boldsymbol{u}}_n = A_n\left(\frac{r}{r_0}\right)^n \sin n\theta \tag{57}$$

where $n = 1, 2, \ldots$, and

$$A_n^2 = \frac{n+1}{\pi r_0^2} \tag{58}$$

It is not difficult to show that the choice $\nabla \cdot \tilde{\boldsymbol{u}}_n = 1$ does not generate another Cosserat eigenfunction.

4. A cylindrical rigid inclusion

If $r_1 = r_0$ and $r_2 = \infty$, the cylindrical shell reduces to a cylindrical rigid inclusion with radius r_0. In this case, only the functions $r^{-n} \cos n\theta$, $r^{-n} \sin n\theta$ are needed. The sets of the discrete Cosserat eigenvalues $\tilde{\omega}_n$, eigenvectors $\tilde{\boldsymbol{u}}_n$ and divergences div $\tilde{\boldsymbol{u}}_n$ are

$$\tilde{\omega}_n = -2 \tag{59}$$

$$\tilde{\boldsymbol{u}}_n = \frac{B_n}{2}\frac{r_0^n}{r^{n-1}}\left(1 - \frac{r_0^2}{r^2}\right)(\cos n\theta e_r + \sin n\theta e_\theta) \tag{60}$$

$$\nabla \cdot \tilde{\boldsymbol{u}}_n = B_n\left(\frac{r_0}{r}\right)^n \cos n\theta \tag{61}$$

and

$$\tilde{\omega}_n = -2 \tag{62}$$

$$\tilde{\boldsymbol{u}}_n = \frac{B_n}{2}\frac{r_0^n}{r^{n-1}}\left(1 - \frac{r_0^2}{r^2}\right)(\sin n\theta e_r - \cos n\theta e_\theta) \tag{63}$$

$$\nabla \cdot \tilde{\boldsymbol{u}}_n = B_n\left(\frac{r_0}{r}\right)^n \sin n\theta \tag{64}$$

where $n = 2, 3, \ldots,$ and

$$B_n^2 = \frac{n-1}{\pi r_0^2} \tag{65}$$

5. Discussion

For a cylindrical shell, we obtained eqn (33) for the discrete spectrum $\tilde{\omega}_n$ and eqn (34) for A_n/B_n. These equations are quadratic, therefore, there are two solution of $\tilde{\omega}_n$, $\tilde{\omega}_n \leqslant -2$ and $\tilde{\omega}_n \geqslant -2$ and two solutions of A_n/B_n, $A_n/B_n < 0$ and $A_n/B_n > 0$, for each value of r_1/r_2. Eqn (32) shows that A_n and B_n should have the same sign if $\tilde{\omega}_n \leqslant -2$, or vice versa.

We rewrite the discrete Cosserat eigenvalues in the form

$$\tilde{\omega}_n^{(i)} = \frac{2}{-1 \pm \sqrt{f_n}} \quad i = 1, 2 \tag{66a}$$

where the positive and negative sign apply when $i = 1$ and $i = 2$, respectively and

$$f_1 = -\frac{1 - (r_1/r_2)^2}{\left[1 + (r_1/r_2)^2\right]\log(r_1/r_2)} \tag{66b}$$

Table 1
The discrete Cosserat spectrum $\tilde{\omega}_n^{(1)}$ for a cylindrical shell

n	$r_1/r_2 = 0.0001$	$r_1/r_2 = 0.001$	$r_1/r_2 = 0.01$	$r_1/r_2 = 0.1$	$r_1/r_2 = 0.2$
1	−2.98287136	−3.22830241	−3.74492566	−5.75461322	−8.24144796
2	−2.00034647	−2.00347011	−2.03524980	−2.41644234	−3.02765208
3	−2.00000006	−2.00000056	−2.00056579	−2.05761916	−2.24391020
4	−2.00000000	−2.00000001	−2.00000775	−2.00769803	−2.06131477
5		−2.00000000	−2.00000010	−2.00097047	−2.01516379
6			−2.00000000	−2.00011715	−2.00364146
7				−2.00001372	−2.00085170
8				−2.00000157	−2.00019508
9				−2.00000018	−2.00004396
10				−2.00000002	−2.00000978
11				−2.00000000	−2.00000215
12					−2.00000047
13					−2.00000010
14					−2.00000002
15					−2.00000000

Table 2
The discrete Cosserat spectrum $\tilde{\omega}_n^{(2)}$ for a cylindrical shell

n	$r_1/r_2 = 0.0001$	$r_1/r_2 = 0.001$	$r_1/r_2 = 0.01$	$r_1/r_2 = 0.1$	$r_1/r_2 = 0.2$
1	−1.50431915	−1.44877212	−1.36430859	−1.21032205	−1.13809393
2	−1.99965365	−1.99654189	−1.96595044	−1.70599415	−1.49318126
3	−1.99999994	−1.99999434	−1.99943453	−1.94551994	−1.80391655
4	−2.00000000	−1.99999999	−1.99999225	−1.99236078	−1.94222754
5		−2.00000000	−1.99999990	−1.99903047	−1.98506272
6			−2.00000000	−1.99988287	−1.99637175
7				−1.99998628	−1.99914902
8				−1.99999843	−1.99980495
9				−1.99999982	−1.99995604
10				−1.99999998	−1.99999022
11				−2.00000000	−1.99999785
12					−1.99999953
13					−1.99999990
14					−1.99999998
15					−2.00000000

$$f_n = (n^2 - 1)\frac{(r_1/r_2)^{2n}\left[(r_1/r_2)^2 + (r_1/r_2)^{-2} - 2\right]}{\left[1 - (r_1/r_2)^{2(n+1)}\right]\left[1 - (r_1/r_2)^{2(n-1)}\right]} \quad n \neq 1 \tag{66c}$$

The discrete Cosserat spectrum for a cylindrical shell is a full spectrum from both $\tilde{\omega}_n^{(1)} \leqslant -2$ and $\tilde{\omega}_n^{(2)} \geqslant -2$. As $n \to \infty$, $\tilde{\omega}_n^{(1)} \to -2$ and $\tilde{\omega}_n^{(2)} \to -2$. There are infinite eigenvalues that approach the condensation point $\tilde{\omega} = -2$. The discrete spectrum $\tilde{\omega}_n^{(1)}$ and $\tilde{\omega}_n^{(2)}$ represented in eqn (66) with different values of r_1/r_2 are calculated to the accuracy 10^{-8}. Results are shown in Tables 1 and 2.

At two limiting cases, $r_1 \to 0$ (solid cylinder) or $r_2 \to \infty$ (cylindrical rigid inclusion), the Cosserat discrete spectrum reduces to a condensation point $\tilde{\omega}_n = -2$ with infinite multiplicity.

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